

Flexible periodic points

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Abstract

We define the notion of ε -flexible periodic point: it is a periodic point with stable index equal to two whose dynamics restricted to the stable direction admits ε -perturbations both to a homothety and a saddle having an eigenvalue equal to one. We show that ε -perturbation to an ε -flexible point allows to change it in a stable index one periodic point whose (one dimensional) stable manifold is an arbitrarily chosen C^1 -curve. We also show that the existence of flexible point is a general phenomenon among systems with a robustly non-hyperbolic two dimensional center-stable bundle.

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1 Introduction

Since the Poincaré's discovery of the transverse homoclinic intersections and the complex behaviors near them, the search of the transverse homoclinic intersections, in other words, the control of the invariant manifolds of the systems, has been one of the central problems in dynamical systems. In the late nineties of last century, a breakthrough was brought by Hayashi's connecting lemma (see [H]). It allows us to control the effect of perturbations on the invariant manifolds and enables us to create new intersections. This perturbation technique provides one basis of recent strong development of study of non-uniformly hyperbolic dynamical systems. For example, [CP] uses connecting lemma and its generalizations for building heterodimensional cycles in order to characterize the robust non-hyperbolic behaviors.

In the uniformly hyperbolic context, the invariant manifold theorem tells us their rigidity. The local stable/unstable manifolds of a uniformly hyperbolic set are discs varying continuously with respect to the points and the variation of the diffeomorphisms. A consequence of this fact is the great constraint of the geometric behavior of invariant manifolds under small perturbations. Such rigidities are paradoxically exploited for the construction of robustly non-hyperbolic systems such as Newhouse's example of robust tangencies (see [N]) or Abraham-Smale's example of robust heterodimensional cycles (see [AS]).

Meanwhile, the control of the variations of stable and unstable manifolds of periodic orbits under small perturbations in non-uniformly hyperbolic systems still remains to be an important issue. On one hand, non-hyperbolic systems in general contain plenty of regions where the local dynamics exhibits hyperbolic behaviors, which also give us some rigidity of the invariant manifolds. On the other hand, we may expect that the absence of uniform hyperbolicity implies the existence of periodic orbits whose invariant manifolds has less rigidity so that it can be altered considerably by small perturbations.

A prototype of such argument can be found in [BD1]. The rescaling invariant nature of C^1 -distance gives a strong freedom for the change of relative position of objects in the homothetic regions, that is, contracting or repelling regions where the diffeomorphism is smoothly conjugated to a homothety. Furthermore, [BD1] gave an example of non-uniformly hyperbolic systems in which the homothetic behaviors are quite abundant. This enables us to construct interesting examples of dynamical systems, such as the construction of *universal dynamics* by the first author with Díaz, or the construction of heterodimensional cycles near the wild homoclinic classes by the second author [S].

The fact that non-hyperbolic dynamics may produce homothetic behavior come back to Mañé argument in [M] for surface diffeomorphism (see also [PS]), and has been generalized in higher dimension in [BB, BDP, BGV]. These works are essentially within the scope of perturbations of the derivative of periodic orbits and the conclusions of them provides us local information of perturbed systems. In this article, we pursue the possibility of such strategy and propose new, semi-local technique of the control of invariant manifolds.

For many applications, we need to control the effect of the perturbation on the invariant manifold, for example to keep an heteroclinic connection under the periodic orbit is changing of index. This is the aim of [G1] where Gourmelon uses invariant cone fields for keeping the strong stable manifold almost unchanged along the perturbation. Here we follow a completely opposite strategy: we will use homothetic region (where there is no strictly invariant cone field) for obtaining a great freedom of choosing the invariant manifold after perturbing the derivative of the periodic orbit.

The aim of this paper is twofold. First, we consider diffeomorphisms of two dimensional manifolds and define the notions of flexible periodic points, which is an abstract sufficient condition on a periodic point such that the above strategy is well available. We investigate the possible perturbation of such points. We extend the concept of flexible points to diffeomorphisms in higher dimension stable index two periodic point and see that the perturbation technique proved in two dimensional setting is valid even in higher dimensional situations. Second, we show that flexible points are quite abundant in some type of higher dimensional partially hyperbolic dynamical systems.

Let us now state our main results.

1.1 Flexible points of surface diffeomorphisms

First, we briefly review the notion of linear cocycles. Let X be a topological space, $f : X \rightarrow X$ be a homeomorphism of X and E be a Riemannian vector bundle over X . A *linear cocycle* on \mathcal{E} is a bundle isomorphism $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}$ which is compatible with f . In this article, we are mainly interested in the situation where f is a diffeomorphism of some manifold, X is a periodic orbit of such dynamical systems and \mathcal{A} is the restriction of differential map acting on the restriction of tangent bundle over the orbit. Such a system can be, by taking coordinate, identified with the situation where $X = \mathbb{Z}/n\mathbb{Z}$ (n is the period of the orbit), $f(x) := x + 1$ and \mathcal{A} is a sequence of regular matrices. We call such system a *linear cocycle over periodic orbit of period n* .

Let \mathcal{A}, \mathcal{B} be linear cocycles on \mathcal{E} over (X, f) . We put

$$\text{dist}(\mathcal{A}, \mathcal{B}) := \sup \| \mathcal{A}(x) - \mathcal{B}(x) \|$$

and call it *distance* between \mathcal{A} and \mathcal{B} ($|x|$ ranges over all unit vectors in all fibers). This defines a topology on the space of linear cocycles on \mathcal{E} . Let \mathcal{A}_t denote a continuous one-parameter family of cocycles, that is, a continuous map from some interval to the space of cocycles. We put

$$\text{diam}(\mathcal{A}_t) := \sup_{s < t} \text{dist}(\mathcal{A}_s, \mathcal{A}_t)$$

and call it *diameter* of \mathcal{A}_t

Now, let us give a precise definition of *flexible cocycles*.

Definition 1. Let $\mathcal{A} = \{A_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, $A_i \in \text{GL}(2, \mathbb{R})$ be a linear cocycle over a periodic orbit of period $n > 0$. Fix $\varepsilon > 0$. We say that \mathcal{A} is ε -flexible if there is a continuous one-parameter family of linear cocycles $\mathcal{A}_t = \{A_{i,t}\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ defined over $t \in [-1, 1]$ such that:

- $\text{diam}(\mathcal{A}_t) < \varepsilon$;
- $A_{i,0} = A_i$, for every $i \in \{0, \dots, n-1\}$;
- A_{-1} is an homothety;
- for every $t \in (-1, 1)$, the product $A_t := (A_{n-1,t}) \cdots (A_{0,t})$ has two distinct positive contracting eigenvalues.

- Let λ_t denote the smallest eigenvalue of the product A_t . Then $\max_{-1 \leq t \leq 1} \lambda_t < 1$.
- A_1 has a real positive eigenvalue equal to 1.

Definition 2. A periodic orbit of a two dimensional diffeomorphism is called ε -flexible if the linear cocycle of the derivatives along its orbit is ε -flexible.

The interesting feature of ε -flexible points is that it has a great freedom for changing the position of their strong stable manifold by an ε -perturbation supported in an arbitrarily small neighborhood of the orbit. More precisely, we can choose the fundamental domain of this strong stable manifold to be any prescribed curve subject to the unique limitation that it should remain in the same isotopy class in the orbit space.

Let us explain that. Let \mathcal{N} be a compact neighborhood of some attracting periodic orbit $\mathcal{O}(x)$ of some diffeomorphism F on two dimensional manifold. Suppose that:

- the derivative of F on $\mathcal{O}(x)$ in the period has two distinct contracting eigenvalues.
- \mathcal{N} is F -invariant and is contained in the basin of $\mathcal{O}(x)$.

Let us consider the *punctured neighborhood* $\mathcal{N} \setminus \mathcal{O}(x)$ and take the quotient space by the orbit equivalence (that is, $x \sim y$ if and only if $F^m(x) = F^n(y)$ for some $m, n \geq 0$). We denote it by T_F^∞ and call it *orbit space*. Then, T_F^∞ is naturally identified with the two dimensional torus \mathbb{T}^2 . This torus is naturally endowed with a homotopy class of a *parallel*, which consists in the class of an essential circle around $\mathcal{O}(x)$. Moreover, the strong stable manifold $W^{ss}(x) \setminus \{x\}$ projects to the quotient space as two parallel circles, cutting the parallel with intersection number 1. We call these two simple closed curves in T_F^∞ *meridians*.

Then, consider another diffeomorphisms G which is a perturbation of F with small support concentrated around $\mathcal{O}(x)$ and preserving the orbit of $\mathcal{O}(x)$. Let $\Lambda_G := \bigcap_{i \geq 0} G^i(\mathcal{N})$ denote the locally maximal invariant set in \mathcal{N} . Now, consider the set in $\mathcal{N} \setminus \Lambda_G$ and take the quotient by the orbit equivalence under G . We denote the orbit space by T_G^∞ . We identify T_F^∞ and T_G^∞ as follows. First, note that the restriction of $G|_{\mathcal{N} \setminus \Lambda_G}$, can be conjugated to $F|_{\mathcal{N} \setminus \mathcal{O}(x)}$ by a unique homeomorphism h which coincides with the identity map outside the support of the perturbation and is C^1 out of $\mathcal{O}(x)$. Let us call this homeomorphism *standard conjugacy*. It gives us an identification between T_F^∞ and T_G^∞ .

Under this identification, the freedom of flexible point mentioned above can be formulated as follows.

Theorem 1. Let f be a C^1 -diffeomorphism of a surface and D be an attracting periodic disc of period π , that is, $D, f(D), \dots, f^{\pi-1}(D)$ are pairwise disjoint and $f^\pi(D)$ is contained in the interior of D . Assume that D is contained in the stable manifold of an ε -flexible periodic point p contained in D . Let $\gamma = \gamma_1 \cup \gamma_2 \subset T_f^\infty$ be the two simple closed curves which $W^{ss}(p)$ projects to.

Then, for any pair of C^1 -curves $\sigma = \sigma_1 \cup \sigma_2$ embedded in T_f^∞ isotopic to $\gamma_1 \cup \gamma_2$, there is an ε -perturbation g , supported in an arbitrarily small neighborhood of p (which is also sufficiently small so that we can define the standard conjugacy) such that g satisfies the following:

- p is a periodic attracting point having a eigenvalue $\lambda_1 \in]0, 1[$ and a eigenvalue $\lambda_2 = 1$.
- D is contained in the basin of p .
- $W^{ss}(p, g)$ projects to $(\sigma_1 \cup \sigma_2) \subset T_g^\infty \simeq T_f^\infty$.

The orbit of p for g is a non-hyperbolic attracting point, having an eigenvalue equal to 1. By an extra, arbitrarily small perturbation, one can change the index of p so that the strong stable manifold becomes the new stable manifold. Therefore we have the following.

Corollary. Under the hypotheses of Theorem 1, there is an ε -perturbation h of f , supported in an arbitrarily small neighborhood of p so that

- p is a periodic saddle point having two real eigenvalues $0 < \lambda_1 < 1 < \lambda_2$.
- $W^s(p) \setminus \mathcal{O}(p)$ is disjoint from the maximal invariant set Λ_H .
- $W^s(p, h)$ projects to $(\sigma_1 \cup \sigma_2) \subset T_h^\infty \simeq T_f^\infty$.

The proof of Theorem 1 is given in Section 2, 3 and 4.

1.2 Stable index two flexible points

The flexible points are certainly not usual for surface diffeomorphisms. However, it appears very naturally in higher dimensional partially hyperbolic systems with two dimensional stable directions. To explain it, first we extend the definition of flexible points in higher dimensional setting and see the direct consequence of Theorem 1 about them. We prepare one notion. Let f be a diffeomorphism of some differential manifold, and p be a (not necessarily hyperbolic) periodic point of it. Then the *stable index* of p , is the number of eigenvalues of the differential of the first return map with absolute value strictly less than one counted with multiplicity.

Definition 1. Let M be a compact manifold endowed with a Riemann metric, f a diffeomorphism of M and $\varepsilon > 0$. A stable index 2 hyperbolic period point x is called ε -flexible if the restriction of Df to the stable direction over $\mathcal{O}(x)$ is an ε -flexible cocycle.

Remark 1. The notion of flexibility is a robust property in the following sense: if q_f is ε -flexible periodic orbit of f then there is a C^1 -neighborhood \mathcal{U} of f so that every $g \in \mathcal{U}$ has a well defined continuation q_g of q_f and q_g is 2ε -flexible.

Let x be a stable index 2 periodic point of period n , with two distinct real positive contracting eigenvalues. For flexible points in this setting, we can also define the notion of orbit space, parallel, and meridians as follows.

Consider the neighborhood \mathcal{N} in the local stable manifold of x which is strictly positively invariant. The orbit space of $\mathcal{N} \setminus \mathcal{O}(x)$ under f is again diffeomorphic to the torus \mathbb{T}^2 and we denote it by T_f^∞ . As is in the previous case, this torus is naturally endowed with parallel and the strong stable manifold of x induce by projection on T_f^∞ two disjoint simple curved called meridians.

We consider perturbation g of f whose support is concentrated to $\mathcal{O}(x)$ and which preserves the forward invariant property of \mathcal{N} . Then, the space of g -orbits in $D \setminus (\Lambda_g)$ (where Λ_g is the locally maximal invariant set of g in \mathcal{N}), denoted by T_g^∞ are naturally identified with T_f^∞ via the standard conjugacy.

It is easy to observe the following:

Remark 2. Any ε -perturbation of the restriction of f to the positive orbit of \mathcal{N} , in a sufficiently small neighborhood of $\mathcal{O}(x)$ can be realized as a ε -perturbation of f . Furthermore, if the perturbation prerserves the periodic orbit \mathcal{N} then one may required that the eigenvalues of x transverse to \mathcal{N} are kept unchanged.

Therefore, as a direct corollary of Theorem 1 we obtain the following:

Corollary 1. Let f be a diffeomorphism of a compact manifold, $\varepsilon > 0$, and x be a ε -flexible stable index 2 hyperbolic periodic point. Fix a strictly forward invariant neighborhood \mathcal{N} of $\mathcal{O}(x)$ contained in the local stable manifold of x . Let T_f^∞ , endowed with the meridians γ_1, γ_2 , denote the orbit space of the punctured stable manifold of x .

Let σ_1 and σ_2 be two disjoint simple curves isotopic to the meridians. then there is an ε -perturbation g of f , supported in an arbitrarily small neighborhood of $\mathcal{O}(x)$, preserving the forward invariance of \mathcal{N} , preserving the orbit $\mathcal{O}(x)$ such that the following holds:

- The eigenvalues in the directions transversal to \mathcal{N} at x of g are same for that of f .
- $\mathcal{O}(x)$ is a stable index 1 saddle point: there is a contracting eigenvalue tangent to \mathcal{N} .

- the punctured stable manifold of x is disjoint from the maximal invariant in \mathcal{N} , and the projection of $W^s(x) \setminus \{x\}$ on T_f^∞ is precisely the two curves $\sigma_1 \cup \sigma_2$.

As a conclusion, for flexible points in this setting, we also have great freedom for the choice of position of strong stable manifolds. Meanwhile, there is a difference. In higher dimensional setting, we have a priori few control on the effect of the perturbation to the local unstable manifold of x , and therefore on the position of the local strong unstable manifold of g .

More precisely, if the angles between the unstable bundle over $\mathcal{O}(x)$ and the orthogonal to the stable bundle is very small all along $\mathcal{O}(x)$, then every ε perturbation of f in the local stable manifold of $\mathcal{O}(x)$, supported in a very small neighborhood of $\mathcal{O}(x)$ and preserving $\mathcal{O}(x)$ may be realized as a ε -perturbation g of f , supported in a small neighborhood of $\mathcal{O}(x)$ and which coincides with f on the local unstable manifold of $\mathcal{O}(x)$. On the other hand, this angle is large, that is, if the angle between the stable and unstable bundles is very small at some point of $\mathcal{O}(x)$, then perturbing f in the stable manifold without perturbing the unstable one can costs a lot.

Such a difference can be a serious problem if one applies this technique to the problem of (non-)existence of homoclinic intersections. However, if we have a priori estimates on the angles mentioned above, the perturbation technique suggested in Corollary 1 works well. Note that such a priori estimates are available in the case where the system admits partially hyperbolic splitting, or more generally, dominated splittings.

1.3 Abundance of flexible points

At a first glance, the definition of flexible points may look strange, as it claims the existence of perturbations to two completely different situations. However, the fact is that such a situation is quite abundant in the context of non-uniformly hyperbolic situations as we will see below.

First let us give one precise statement in the form of a C^1 -generic property.

Theorem 2. *There is a residual subset $\mathcal{G} \subset \text{Diff}^1(M)$ so that for every $f \in \mathcal{G}$, for any chain recurrence class C containing*

- a periodic point p of stable index 2 with complex (non real) contracting eigenvalue
- a periodic point q of stable index 1

and for any $\varepsilon > 0$ there are ε -periodic point p_n homoclinically related to p and whose orbits γ_n converges to the chain recurrence class C in the Hausdorff topology.

Let us see that there are large class of diffeomorphisms satisfying the hypotheses of Theorem 2. To explain it, let us briefly review the notion of robust heterodimensional cycles. We say that two hyperbolic basic sets K and L of a diffeomorphisms f form a C^1 -robust heterodimensional cycle if

- the stable-indices of K and L are different,
- for any g sufficiently C^1 -close to f the continuations K_g and L_g of K and L satisfies that

$$W^u(K_g) \cap W^s(L_g) \neq \emptyset \text{ and } W^s(K_g) \cap W^u(L_g) \neq \emptyset.$$

Robust heterodimensional cycles are very important mechanisms for the study of robustly non-hyperbolic behaviors of diffeomorphisms, as they are the mechanisms that account for the birth of robust non-hyperbolicity in the large class of C^1 non-hyperbolic diffeomorphisms. Indeed, up to now, all the known examples of robustly non-hyperbolic behaviors are ascribed to robust heterodimensional cycles, and it is conjectured by the first author in [B] that every robustly non-hyperbolic diffeomorphisms could be approximated by one which has a robust heterodimensional cycles. Furthermore, it is worth mentioning that the creation of robust heterodimensional cycle is a quite general phenomenon from the bifurcation of heterodimensional cycles between saddles of different indices (see [BD2] and [BDK]).

Then, let us consider diffeomorphisms satisfying following conditions:

- It has a robust heterodimensional cycle between two hyperbolic basic sets K and L .
- K has stable index one and L has stable index two.
- L has a periodic point with complex eigenvalues to the stable directions.

The set of such diffeomorphisms forms, by definition, an open set in $\text{Diff}^1(M)$ and non-empty if $\dim M \geq 3$. Then, every diffeomorphism contained in the intersection of this open set and the residual set in Theorem 2 serves as an example (for the chain recurrence set take the one that contains K and L).

To observe the largeness of the class of diffeomorphisms which are within the range of hypotheses of Theorem 1, let us briefly discuss the relationship between the hypotheses of Theorem 1 and the notion of *homoclinic tangencies*. The hypotheses of Theorem 1 requires the existence of periodic point which has complex eigenvalues to the stable direction. This implies the indecomposability of the stable directions. The other condition guarantees that the stable direction is not uniformly contracting. By the work of Gourmelon [G2] and Wen [W], these condition is equivalent to that this diffeomorphism can be approximated by the one with homoclinic tangency of stable index one. Remember that, the result [BD2] tells that the class of diffeomorphisms which satisfies such conditions are quite large.

The proof of Theorem 2 is a consequence of Theorem 3 below combined with the generic property that C^1 -generically a homoclinic class coincides with the chain recurrence class which contains it (see [BC]) and the coexistence of the periodic point of different indices implies the existence of robust heterodimensional cycles [BD2]. To state Theorem 3, we prepare some definitions. Given a periodic point p and a neighborhood U of p , the *relative homoclinic class* $H(p, U)$ of p in U is the closure of the set of transverse homoclinic points whose whole orbit is contained in U . A periodic point q is *homoclinically related with p in U* if there are transverse intersection orbits which is contained in U and going from p to q and from q to p .

Now, Theorem 2 is a consequence of the following "local" result:

Theorem 3. *Let f be a diffeomorphisms of a compact manifold. Suppose that f admits a hyperbolic periodic point p and an open neighborhood U of the orbit $\mathcal{O}(p)$ with the following properties:*

- *p has stable index equal to 2.*
- *there is periodic point p_1 homoclinically related with p in U , so that the p_1 has a complex (non-real) contracting eigenvalues.*
- *there is a periodic point q with $\mathcal{O}(q) \subset U$ with stable index 1.*
- *there are hyperbolic transitive basic sets $K \subset H(p, U)$ and $L \subset H(q, U)$ containing p and q , respectively, so that K, L form a C^1 -robust heterodimensional cycle in U .*

Then, for any $\varepsilon > 0$ there are arbitrarily small C^1 -perturbations g of f having a ε -flexible point homoclinically related with p_g in U and which orbit is ε -dense in the relative homoclinic class $H(p_g, U)$.

We give the proof of Theorem 3 (thus also Theorem 2) in Section 5.

1.4 Possible dynamical consequences and generalizations

The notion of flexible points and its abundance are interesting in themselves. At the same time, we think that it would be a powerful tool for the study of C^1 -generic systems in many ways. Let us explain some idea with some details.

The first possible application is for the investigation of *tame/wild* properties in $\text{Diff}^1(M)$ (see [B]). In a further work, the authors will use them as a mechanism for producing new example of wild diffeomorphisms, that is, C^1 -generic diffeomorphisms with infinitely many chain recurrence classes. The idea is very simple: if p is a flexible point of stable index 2, one can transform p to be a stable index 1 periodic point whose stable manifold is an arbitrarily chosen curve in the old 2-dimensional stable manifold. If we can choose this curve being disjoint from the initial chain recurrence class, then

it implies that the point has been ejected from the class. Repeating this procedure, we obtain infinitely many saddles with trivial homoclinic classes in a neighborhood of any classes satisfying hypothesis of Theorem 2.

However, this strategy is not complete: there are C^1 -robustly transitive diffeomorphisms which satisfies the hypothesis of Theorem 2 (see for example [BV]). They have plenty of flexible points, but cannot be expelled to outside the class! The reason is that, since the class being the whole manifold, there is no space to escape from the original classes. Thus, to execute this strategy completely, we need to investigate the *topology of the chain recurrence class* looked inside the center stable directions, which will be one of the central topic in [BS].

We suggest another possible direction of application. The control of the position of the stable manifold may open the way to study the difference between C^1 -generic diffeomorphisms and the C^1 -open diffeomorphisms. More precisely, for example, it is known that for C^1 -generic robustly transitive diffeomorphisms, the homoclinic class of every hyperbolic periodic points coincides with the whole manifold (this is a consequence of Hayashi's connecting lemma). One interesting question is to see if this property is an open property. A priori, there is no reason that it is. Meanwhile, to establish a rigorous conclusion is not such a simple matter. For that, what we need to understand is the position of homoclinic intersections. Under the situation where we have abundance of flexible points, obtaining the better understanding of the position of homoclinic intersections sounds quite feasible, since the perturbation technique of Theorem 1 provides us with a strong freedom for the control of the (un)stable manifolds.

In this paper, we define the notion of flexible points of stable index 2. We can define similar notions for higher stable f cases, for example, as points whose derivative to stable direction can be perturbed both to a contracting homothety and to a saddle having at least one eigenvalue equal to 1. It would be interesting to establish similar perturbation techniques to choose control the position of stable manifolds for them, and to study possible topology of flags of strong stable manifolds in the orbit spaces. However, creating a deformation of a linear cocycle in higher dimension is much more difficult and technical than in dimension 2. Therefore, we think it is better to restrict our attention to two dimensional situation, and leave the generalization to higher dimensional cases by the time when we will be well prepared for such ambitious researches.

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2 The flexibility and the control of the stable manifold

The aim of section 2, 3 and 4 is the proof of Theorem 1. It consists of noticing that the flexibility property which allows us to perform an ε -perturbation of the flexible linear cocycle, among the cocycle of diffeomorphisms, which inserts a region where the dynamics in the period is a homothety. Furthermore, the number of fundamental domains in the homothetic region may be chosen arbitrarily. In some sense, we ask the orbit to lose an arbitrarily large time in that homothetic region. For this reason we call them *retardable cocycle*.

Iterating a homothetic region does not introduce any distortion: as we will see, it is therefore easy to control the effect of perturbations performed inside the homothetic region. The fact that we may use an arbitrary number of fundamental domains leaves us the time to deform slowly the strong unstable manifold to the a priori chosen curve.

2.1 Retardable cocycle

To state the meaning of “inserting a lot of homothetic regions” we first define the notion of retardable cocycles. Let us give the notion of diffeomorphism cocycles over a finite orbit. We consider cocycles of diffeomorphisms on $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, for which $\{\mathbf{0}_i\}$ (where we put $\mathbf{0}_i := (0, i)$) is a periodic sink attracting all the points. More precisely, a *diffeomorphism cocycle* is a set of diffeomorphisms $\mathcal{F} =$

$\{f_i \mid \mathbb{R}^2 \times \{i\} \rightarrow \mathbb{R}^2 \times \{i+1\}\}$. We denote the map induced on the total space $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ from the cocycle $\mathcal{F} = \{f_i\}$ also by \mathcal{F} .

In this article, we assume that every diffeomorphism cocycle fixes the origin, that is, we always assume $f_i(\mathbf{0}_i) = \mathbf{0}_{i+1}$ for every i . Such cocycle is called *contracting* if the 0-section is an attracting periodic orbit and if any orbits is contained in its basin. Given a linear cocycle $\mathcal{A} = \{A_i\}$, we consider it as a diffeomorphism cocycle in the obvious way, that is, we consider A_i to be the diffeomorphism from $\mathbb{R}^2 \times \{i\}$ to $\mathbb{R}^2 \times \{i+1\}$. For a diffeomorphism cocycle $\mathcal{F} = \{f_i\}$, we denote the first return map of it on \mathbb{R}^2 by F (drop the subscript in $\mathbb{Z}/n\mathbb{Z}$ and capitalize the symbol.) Note that a linear cocycle $\mathcal{A} = \{A_i\}$ is contracting if and only if all eigenvalues of A have absolute value strictly less than one.

In the following, we denote the two dimensional disk of radius r centered at $\mathbf{0}_i$ by $B_i(r) \subset \mathbb{R}^2 \times \{i\}$ and for any $0 < r < s$ we denote by $\Gamma_{r,s}$ the round closed annulus in $\mathbb{R}^2 \times \{0\}$ bounded by the circles of radii r and s , that is, $\Gamma_{r,s} := \overline{B_0(s)} \setminus B_0(r)$.

Definition 3. A contracting cocycle of diffeomorphisms $\mathcal{F} = \{f_i\}$ over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ is called *retardable* cocycle if it satisfies the following conditions:

There exists R_1, R_2, R_3 satisfying $0 < R_1 < R_2 < R_3$ such that

- $f_i|_{B_i(R_1) \setminus B_i(R_3)} = A_i$, where A_i is a linear map such that $A = \prod_{j=0}^{n-1} A_j = \lambda \text{Id}$ where $0 < \lambda < 1$. In other words, A is a contracting homothety.
- For every $x \in B_0(R_2)$ and i satisfying $0 \leq i < \pi$, we have $(\prod_{j=0}^{i-1} f_j)(x) \in B_i(R_1)$.
- $A(B_0(R_2))$ contains $B_0(R_3)$ in its interior. We call $\Gamma_{\lambda R_2, R_2}$ homothetic region of \mathcal{F} .

The main property of retardable cocycles is that one may insert arbitrarily many fundamental domains of homothety as follows:

Proposition 1. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ be a retardable cocycle. We define new cocycles $\mathcal{F}_m = \{f_{i,m} \mid \mathbb{R}^2 \times \{i\} \rightarrow \mathbb{R}^2 \times \{i+1\}\}$, ($m \geq 0$) as follows:

- For $x \in \mathbb{R}^2 \times \{i\}$ with $|x| > R_3$, $f_{i,m}(x) := f_i(x)$.
- For $x \in \mathbb{R}^2 \times \{i\}$ with $\lambda^m R_3 < |x| < R_3$, $f_{i,m}(x) := A_i(x)$.
- For $x \in \mathbb{R}^2 \times \{i\}$ with $|x| \leq \lambda^m R_3$, $f_{i,m}(x) := (A^m \circ f_i \circ A^{-m})(x)$.

Then those maps define a C^1 -diffeomorphisms contracting cocycle. We call $\{f_{i,m}\}$ the m -retard of $\{f_i\}$.

The proof of above proposition is obvious, so we omit it.

Roughly speaking, $\{f_{i,m}\}$ is a cocycle that is obtained by *insertion of m -homothetic fundamental domains* to $\{f_i\}$. This insertion does not change the main dynamical properties of the cocycle: the orbits will just spend more time in the newly added homothetic region. More precisely, on the homothetic region, the relative position of objects such as the strong stable manifold are kept intact under the iteration. For example, we have the following properties of retarded cocycles:

Remark 3. • $\{f_{i,0}\} = \{f_i\}$.

- Let $\Gamma_{\lambda R, R}$ be the homothetic region of \mathcal{F} . Then, $F|_{\Gamma_{\lambda^{m+1}R, R}}$ is a homothety of rate of contraction λ . We call $\Gamma_{\lambda^{m+1}R, R}$ homothetic region of \mathcal{F}_m .
- Suppose the origin $\{\mathbf{0}_i\}$ has strong stable manifold $W^{ss}(\mathbf{0}_0, \mathcal{F})$ of $\{f_i\}$. Then, we have

$$W^{ss}(\mathbf{0}_0, \mathcal{F}_m) \cap \Gamma_{\lambda^{l+1}R, \lambda^l R} = A^l(W^{ss}(\mathbf{0}_0, \mathcal{F}) \cap \Gamma_{\lambda R, R})$$

for l satisfying $0 \leq l \leq m$.

- The item above can be stated more sophisticated way by using the language of orbit spaces. Note that for every $m \geq 0$, $\mathcal{F}_m = \{f_{i,m}\}$ coincides with $\mathcal{F} = \{f_i\}$ outside some compact neighborhood of the origin. Thus the standard conjugacy gives the natural identification between $T_{\mathcal{F}_m}^\infty$ and $T_{\mathcal{F}}^\infty$. Then, the above item is paraphrased that $W^{ss}(\mathbf{0}_0, \mathcal{F}_m)$ and $W^{ss}(\mathbf{0}_0, \mathcal{F})$ project to the same curve in $T_{\mathcal{F}_m}^\infty = T_{\mathcal{F}}^\infty$.

Furthermore, under some special circumstance, the operation of retarding does not change the dynamics so much. To explain that, we introduce the notion of distance on diffeomorphism cocycles. Let $\{f_i\}, \{g_i\}$ be two diffeomorphism cocycle on $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$. We say that $\{g_i\}$ is a *perturbation* of $\{f_i\}$ if the support, that is, the set $\{x \in \mathbb{R}^2 \times \{i\} \mid f_i(x) \neq g_i(x)\}$ is relatively compact for all i . Suppose $\{g_i\}$ is a perturbation of $\{f_i\}$ and $\varepsilon > 0$. Then $\{g_i\}$ is called ε -perturbation of $\{f_i\}$ if

$$\max_{i \in \mathbb{Z}/n\mathbb{Z}, x \in \mathbb{R}^2 \times \{i\}} \|Df_i(x) - Dg_i(x)\| < \varepsilon.$$

For the notion of ε -perturbation of diffeomorphism cocycle, see Remark 4 below. Then, for the retarded cocycle, we have the following.

Lemma 1. *Suppose $\{A_i\}$ is a linear cocycle and $\{f_i\}$ is a retardable diffeomorphism cocycle which is also an ε -perturbation of $\{A_i\}$ such that the support of the perturbation and the homothetic region is contained in a neighborhood \mathcal{N} of $\{\mathbf{0}_i\}$. Then $\{f_{i,m}\}$ is also a ε -perturbation of $\{A_i\}$ whose support is contained in \mathcal{N} .*

Remark 4. *The notion of ε -perturbation gives a concept of closeness between a cocycle and its perturbation. A priori, this is different from the notion of usual C^1 -distance, since our one does not take the contribution of C^0 -distance into consideration. However, this difference is negligible for the following reason: In the following, we establish a perturbation technique which provides us an ε -perturbation with very small (indeed, arbitrarily small) support. This smallness of support combined with the smallness of the ε implies the smallness of C^0 -distance and supplements the lack of it in the practical use.*

By inserting a lot of homothetic regions, combining the fragmentation lemma, we will see that we can obtain a considerably large freedom to change the relative position of the objects.

2.2 Proof of Theorem 1

The aim of this section is to prove Theorem 1 as a consequence of the following propositions.

The first proposition makes the relation between flexible and retardable cocycles: retardable cycles may be obtained as small perturbations of flexible cocycles.

Proposition 2. *Let $\varepsilon > 0$ and $\mathcal{A} = \{A_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, $A_i \in \text{GL}(2, \mathbb{R})$ be an ε -flexible linear cocycle over a periodic orbit of period $n > 0$.*

Then there is a contracting retardable diffeomorphisms cocycle $\mathcal{F} = \{f_i\}$ with the following properties:

- *For any $m \in \mathbb{N}$, the retarded cocycle $\mathcal{F}_m = \{f_{m,i}\}$ is an ε -perturbation of \mathcal{A} .*
- *There is an isotopy of contracting diffeomorphism cocycle connecting \mathcal{F} and \mathcal{A} such that for every moment the periodic orbit $\{\mathbf{0}_i\}$ has two different real eigenvalues.*
- *For every $i \in \mathbb{Z}/n\mathbb{Z}$, the map f_i coincides with A_i out of the unit balls $B_i(1) \subset \mathbb{R}^2 \times \{i\}$.*
- *The derivative DF at the origin $\{\mathbf{0}_0\}$ has a contracting eigenvalue and one eigenvalue equal to 1.*

The second proposition explores the effect of perturbations of retardable cocycles on the position of the strong stable manifold.

Proposition 3. *Let $\mathcal{F} = \{f_i\}$ be a retardable diffeomorphism cocycle over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ whose origin has two distinct real positives eigenvalues. For any pair of disjoint simple curves σ_1, σ_2 in $T_{\mathcal{F}}^\infty$, isotopic to the meridians, and for any $\varepsilon_0 > 0$, there is $N > 0$ so that, for every $m \geq N$ there is a ε_0 -perturbation \mathcal{G} of the m -retarded cocycle \mathcal{F}_m such that the following holds:*

- \mathcal{G} is a perturbation of \mathcal{F}_m with support in the homothetic region.
- \mathcal{G} is a contracting cocycle.
- The strong stable manifold of the origin $\{0_i\}$ induces $\sigma_1 \cup \sigma_2$ on the orbit space $T_{\mathcal{G}}^\infty$.

Let us give the proof assuming these two perturbation results.

Proof of Theorem 1 using Propositions 2 and 3. Let f be a C^1 -diffeomorphism of some surface and D be an attracting periodic disc, in the basin of an ε -flexible hyperbolic periodic point p of period n . Remember that the orbit space in the punctured stable manifold of p is a torus T_f^∞ endowed with a parallel and a meridian (isotopy class of the projection of the strong stable manifold of p).

First, we perform a perturbation along the orbit of p so that we can reduce the problem to the linear cocycle case, which enables us to use Proposition 2 and 3. In the following, all the perturbations we give are tacitly assumed to be supported in sufficiently small neighborhood of the orbit of p so that the orbits entering in D can always be identified with a point to T_f^∞ by standard conjugacy. We fix a pair σ_1, σ_2 of disjoint simple curves isotopic to a meridian. Then, by an arbitrarily C^1 -small perturbation of f supported in an arbitrarily small neighborhood of the orbit of p , one can obtain a diffeomorphism f_0 whose expression in local charts around the orbit of p is linear, and coincides with the differential of f along the orbit of p . Furthermore, f_0 is isotopic to f through cocycles with the same eigenvalues along the orbit of p , so that the continuous dependance of the strong stable manifolds implies that the meridian of f_0 in $T_{f_0}^\infty$ are isotopic to the meridian of f .

Therefore, by changing f with f_0 , we can assume that f is linear in a neighborhood of the orbit of p . Let $\mathcal{A} = \{A_i = Df(f^i(p))\}$ be the corresponding linear cocycle. As f is linear near the orbit of p and is a contraction, the diffeomorphism f is C^1 -conjugated to \mathcal{A} by a unique diffeomorphism which is the identity map in the small neighborhood of the orbit of p . Thus the space of orbits of the punctured linear cocycle $T_{\mathcal{A}}^\infty$ is canonically identified to T_f^∞ , via this conjugacy. Therefore, the circles σ_1 and σ_2 of T_f^∞ induce circles α_1 and α_2 of $T_{\mathcal{A}}^\infty$. Now, the problem is translated to the perturbation problem of linear cocycles \mathcal{A} over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, that is, for proving Theorem 1, we only need to show that there are ε -perturbations of \mathcal{A} in an arbitrary small neighborhood of the orbit of $\{0_i\}$ which satisfies conclusion of the theorem for the curves for α_1 and α_2 in $T_{\mathcal{A}}^\infty$. Let us construct such perturbation by using Proposition 2 and 3.

Proposition 2 allows us to perform an ε -perturbation of \mathcal{A} in order to get a retardable cocycle \mathcal{F} which coincides with \mathcal{A} outside the unit ball. By conjugating this perturbation by an homothety (which does not change the C^1 -size of the perturbation), we can assume that the support of the perturbation given by Proposition 2 is contained in an arbitrarily small neighborhood of the orbit of the origin. Remember that the orbit of the origin for \mathcal{F} and hence for \mathcal{F}_m is a non-hyperbolic attracting orbit having exactly one eigenvalue equal to 1, as announced in Theorem 1. Thus we can talk about the meridians in $T_{\mathcal{F}}^\infty$. Since \mathcal{F} is isotopic to \mathcal{A} through contracting cocycle having distinct real eigenvalues supported in the small ball, we have that the meridians of \mathcal{F} in $T_{\mathcal{F}}^\infty$ are isotopic to the meridians of \mathcal{A} . Remark 3 tells us that the same holds for the m -retarded cocycles \mathcal{F}_m .

Now we apply Proposition 3: for m large enough, \mathcal{F}_m admits an arbitrarily C^1 -small perturbation supported in the homothetic region, so that the strong stable manifold of the periodic orbit induces the circles α_1 and α_2 on the orbit space $T_{\mathcal{A}}^\infty = T_{\mathcal{F}}^\infty = T_{\mathcal{F}_m}^\infty$. \square

It remains to prove Propositions 2 and 3. In Section 3 we prove Propositions 2 and In Section 4 we prove 3.

3 Perturbation of retardable cocycles in the homothetic region

In this section we will prove Proposition 3. The idea, already appeared in [BD1] and [BCVW], is to combine two simple ideas:

- The fragmentation lemma, which asserts that every diffeomorphism of a closed manifold isotopic to the identity map can be written as a finite product of diffeomorphisms arbitrarily close to the identity map, supported in balls with arbitrarily small balls.
- Conjugating a diffeomorphism supported in a small ball by a contracting homothety does not change its C^1 distance to the identity. Therefore if one considers an ε -perturbation of a retarded cocycle \mathcal{F}_m supported in some fundamental domain in the homothetic region, and if we put this perturbation in another fundamental domain by conjugating it by a homothety, it remains an ε -perturbation.

Note that the second item is one of the main idea of Franks' Lemma (linearization of local dynamics near the periodic point by arbitrarily small perturbation, see [F]), which is frequently used in the study of C^1 -generic dynamical systems.

Let us start the proof.

Proof of Proposition 3. Let $\mathcal{F} = \{f_i\}$ be a retardable contracting cocycle over $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$, and $T_{\mathcal{F}}^{\infty}$ the space of orbit of the punctured cocycle. Let $\gamma_1, \gamma_2 \subset T_{\mathcal{F}}^{\infty}$ be the meridian induced by the strong stable manifold of \mathcal{F} . Remember that for any contracting cocycle \mathcal{G} which coincides with \mathcal{F} , the punctured orbit space $T_{\mathcal{G}}^{\infty}$ is identified with $T_{\mathcal{F}}^{\infty}$ through standard conjugacy. In particular, according to remark 3, we have that $W^{ss}(\mathbf{0}_0, \mathcal{F}_m)$ (remember that $\mathcal{F}_m = \{f_{i,m}\}$ is m -retarded cocycle of \mathcal{F}) projects to the same meridian in $T_{\mathcal{F}}^{\infty} = T_{\mathcal{F}_m}^{\infty}$ for every $m \in \mathbb{N}$.

Let $\sigma_1, \sigma_2 \subset T_{\mathcal{F}}^{\infty}$ be two disjoint simple curves which are isotopic to meridians. We take a diffeomorphism $\psi: T_{\mathcal{F}}^{\infty} \rightarrow T_{\mathcal{F}}^{\infty}$ which is isotopic to the identity and satisfies $\psi(\gamma_i) = \sigma_i$ for $i = 1, 2$. Fix some $\varepsilon_0 > 0$. For proving Proposition 3 one has to show that there is N so that every \mathcal{F}_m with $m \geq N$ admits a ε_0 -perturbation supported in the homothetic region, and so that the corresponding meridian are the σ_i .

We want to perturb \mathcal{F}_m in the homothetic region to realize the behavior of φ . For that let us first consider the relation between the diffeomorphism on the orbit space and that of original space. Consider a diffeomorphism $\varphi: \mathbb{R}^2 \times \{0\} \rightarrow \mathbb{R}^2 \times \{0\}$ whose support is contained in a fundamental domain of the return map $F_m: \mathbb{R}^2 \times \{0\} \rightarrow \mathbb{R}^2 \times \{0\}$ (remember that F_m denotes the first return map of diffeomorphism cocycle \mathcal{F}_m). Then φ projects to a diffeomorphism of $T_{\mathcal{F}_m}^{\infty}$. Let us denote this projection by $\tilde{\varphi}$.

In some special case, we can also define the lift of the diffeomorphisms on $\mathbb{R}^2 \times \{0\}$ to $T_{\mathcal{F}_m}^{\infty}$. To describe it, we prepare one notion. The round circles centered at the origin in $\mathbb{R}^2 \times \{0\}$ contained in the homothetic region of \mathcal{F}_m induce a foliation by parallels on $T_{\mathcal{F}_m}^{\infty}$. We call each leaf of this foliation a *round parallel*. Then, we have the following (remember that $\Gamma_{s,t}$ in the claim denotes the annulus bounded by two circles centered at the origin with radii $0 < s < r$).

Claim 1. *Given any $\eta > 0$ there is $\mu > 0$ satisfying the following: Let $\tilde{\varphi}$ be a diffeomorphism of $T_{\mathcal{F}_m}$ which satisfies*

- *The C^1 -distance between $\tilde{\varphi}$ and the identity map is less than μ .*
- *There is a round parallel disjoint from the support of $\tilde{\varphi}$.*

Then for any $m > 0$ and any r so that $\Gamma_{\lambda^2 r, r}$ is contained in the homothetic region of \mathcal{F}_m , there exists a diffeomorphism φ , supported in a round fundamental domain contained in $\Gamma_{\lambda^2 r, r}$, whose projection on $T_{\mathcal{F}_m}^{\infty}$ is $\tilde{\varphi}$ and is an η -perturbation of the identity map (for the definition of C^1 -distance on diffeomorphisms of $T_{\mathcal{F}_m}^{\infty}$, see Remark 5).

Proof. The fact that the support of $\tilde{\varphi}$ is disjoint from one round parallel implies that it admits a lift on some round fundamental domain $\Gamma = \Gamma_{\lambda r_0, r_0} \subset \mathbb{R}^2 \times \{0\}$ of F_m in a homothetic region. Up to some homothetic conjugacy we can assume that $\Gamma \subset \Gamma_{\lambda^2 r, r}$.

Under this situation, one can easily see that there exists a (unique) lift $\tilde{\varphi}$ to $\mathbb{R}^2 \times \{0\}$ supported in Γ . Let us consider the C^1 -distance between the identity map and φ . Since the C^1 -distance of φ to the identity does not depend on the choice of the lift in the homothetic region we only need to consider a specific lift in a $\Gamma_{\lambda^2 r, r}$. Since this correspondence $\tilde{\varphi} \mapsto \varphi$ is continuous and it sends the identity map on T_F to the identity map on $\mathbb{R}^2 \times \{0\}$, the choice of small μ endorses the closeness of the lifted diffeomorphism to the identity map. \square

We perform a perturbation by composing such lifted maps to \mathcal{F}_m . Let us see the effect of such perturbation. First, for the C^1 -distance, we have the following: given φ supported in a round fundamental domain contained in the homothetic region of F_m , we denote by $\mathcal{F}_{m, \varphi} := \{f_{i, m, \varphi}\}$ the perturbation of the cocycle F_m defined by $f_{i, m, \varphi} := f_{i, m}$ if $i \neq n-1$ and $f_{n-1, m, \varphi} := \varphi \circ f_{n-1, m}$. Then there is C (depends only on $f_{n-1, m}$) so that for every m , every $\eta > 0$ and every φ which is η -perturbation of the identity map, then $\mathcal{F}_{m, \varphi}$ is a $C\eta$ -perturbation of the cocycle \mathcal{F}_m .

For the behavior of the strong stable manifold, we have the following (Lemma 2 below follows immediately from the definition):

Lemma 2. *Let $0 < r_1 < \lambda r_2 < r_2 < \lambda r_3 \cdots < r_k$ and $m > 0$ given so that the round annulus $\Gamma_{\lambda r_1, r_k}$ is contained in the homothetic region of \mathcal{F}_m . Let $\{\varphi_i\}$ ($i = 1 \dots, k$) be diffeomorphisms on $\mathbb{R}^2 \times \{0\}$ such that φ_i is supported in $\Gamma_i = \Gamma_{\lambda r_i, r_i}$, and let Φ be the diffeomorphism which coincides with φ_i on Γ_i and equal to the identity outside $\Gamma_{\lambda r_1, r_k}$. Then we have the following:*

- $\mathcal{F}_{m, \Phi}$ is a contracting cocycle which coincides with \mathcal{F}_m out of the homothetic region.
- The meridians of $\mathcal{F}_{m, \Phi}$ are $(\tilde{\varphi}_k)^{-1} \circ \dots \circ (\tilde{\varphi}_1)^{-1}(\gamma_i)$

Now let us perform the perturbation. Consider $\eta < \varepsilon_1/C$ and μ associated to η by Claim 1. The fragmentation lemma ensures that the diffeomorphism ψ (for which $\psi(\gamma_i) = \sigma_i$) can be written as

$$\psi = (\tilde{\varphi}_k)^{-1} \circ \dots \circ (\tilde{\varphi}_1)^{-1}$$

where $k > 0$ and $\tilde{\varphi}_i$ are diffeomorphisms of T_F supported in small discs (so that each φ_i has at least one round parallel disjoint from its support) and so that the C^1 -distance from the identity is less than μ .

Then we fix $m > 3(k+1)$ so that there is a round annulus $\Gamma_{\lambda^m r, r}$ contained in the homothetic regions of \mathcal{F}_m . Then, each $\tilde{\varphi}_i$ admits a lift φ_i supported in an annulus $\Gamma_i = \Gamma_{\lambda r_i, r_i}$ for some $\lambda^{m-3i+1}r \leq r_i \leq \lambda^{m-3i}r$ such that the sequence $\{r_i\}$ satisfies the hypotheses of Lemma 2. Therefore, $\mathcal{F}_{m, \Phi}$ is the announced ε_1 -perturbation where Φ is the diffeomorphism which coincides with φ_i on Γ_i and equal to the identity out of the Γ_i . \square

Remark 5. *In Claim 1, we did not specify the definition of C^1 -distance put on the space of C^1 -diffeomorphisms on T_F . In fact, as was elucidated in the proof, such a choice is not important for Claim 1 and the whole proof.*

4 Construction of retardable cocycles

The aim of this section is the proof of Proposition 2. Let $\mathcal{A} = \{A_i\}$ be an ε -flexible contracting linear cocycle, $i \in \mathbb{Z}/n\mathbb{Z}$. This gives us, by definition, a path $\{A_{i,t}\}$ of contracting linear cocycles. We will use this path for building a retardable contracting cocycle isotopic to \mathcal{A} with several other properties. Our main tool is the Proposition 4 below, which realizes paths of contracting linear cocycles as diffeomorphisms contracting cocycles. Note that Proposition 4 is independent of the notion of flexibility.

We prepare one notation. Let

$$\mathcal{C}_n := \text{GL}(2, \mathbb{R})^n = \{\mathcal{A} = \{A_i\} \mid A_i \in \text{GL}(2, \mathbb{R}), i \in \mathbb{Z}/n\mathbb{Z}\}$$

be the space of linear cocycles of period n . Remember that a cocycle is called contracting if the total space $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ is contained in the basin of the orbit of the origin. We denote by $\mathcal{C}_{n,\text{con}} \subset \mathcal{C}_n$ the (open) subset of contracting cocycles.

Proposition 4. *Let $\mathcal{O} \subset \mathcal{C}_{n,\text{con}}$ be a relatively compact open subset. Then for any $\varepsilon_1 > 0$ there is $\delta > 0$ with the following property: Consider any C^1 -path $\mathcal{A}_t : [0, +\infty[\rightarrow \mathcal{C}_n$, $t \mapsto \{A_{i,t} \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ which is constant near $t = 0$. Assume that we have*

$$\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t}.$$

Then the cocycle of maps $\mathcal{F} = \{f_i \mid \mathbb{R}^2 \times \{i\} \rightarrow \mathbb{R}^2 \times \{i\}, i \in \mathbb{Z}/n\mathbb{Z}\}$ defined as

$$f_i(x) := A_{i,\|x\|}(x),$$

satisfies the following:

- \mathcal{F} is a contracting diffeomorphisms contracting cocycle;
- At each point (x, i) one has

$$\|Df_i(x) - A_{i,\|x\|}\| < \varepsilon_1.$$

Let us first show how to derive Proposition 2 from Proposition 4.

4.1 Proof of Proposition 2

Let $\mathcal{A} = \{A_i\}$ be a ε -flexible cocycle and $\mathcal{A}_t = \{A_{i,t}, t \in [-1, 1]\}$ be the path of cocycles in the definition of the ε -flexibility. Proposition 2 claims the existence of a contracting diffeomorphisms cocycle \mathcal{F} coinciding with \mathcal{A} out of the unit ball, with the homothety on the n first iterates of a round fundamental domain, and with A_1 in a neighborhood of the orbit of the origin. Furthermore, \mathcal{F} needs to be isotopic to \mathcal{A} through contracting cocycles coinciding with \mathcal{A} out of the unit ball and having two different real positive eigenvalues.

Recall that \mathcal{A}_t is a contracting cocycle for $t \neq 1$, but \mathcal{A}_1 is not a contracting eigenvalue. In order to deal with contracting property we will show the proposition but replacing \mathcal{A}_1 by $\mathcal{A}_{1-\eta}$ for some very small η : the corresponding diffeomorphisms cocycle will coincide with $\mathcal{A}_{1-\eta}$ in a neighborhood of the periodic orbit and an extra small perturbation will change $\mathcal{A}_{1-\eta}$ in \mathcal{A}_1 .

The path $\mathcal{A}_t, t \in [-1, 1 - \eta]$ is a compact segment in the open set of contracting linear cocycle. Therefore we can approximate it by a smooth path with the same properties: Thus we assume that $t \mapsto \mathcal{A}_t$ is smooth. Recall that \mathcal{A}_{-1} is a homothety of ratio $\lambda < 1$ in the period, $\mathcal{A}_0 = \mathcal{A}$, and \mathcal{A}_t has two different real eigenvalues for $t \neq -1$. First, we reparametrize \mathcal{A}_t in order to apply Proposition 4.

Lemma 3. *Let $a(t) : [0, 1] \rightarrow V$ be a smooth path in an Euclidean space V . Then, for every $\delta > 0$ there exists a smooth non-decreasing function $\theta : [0, +\infty) \rightarrow [0, 1]$ satisfying the following:*

- $\theta(t) \equiv 0$ near $t = 0$.
- $\theta(t) \equiv 1$ for $t > 1$.
- For every $t \in (0, +\infty)$, we have the following inequality:

$$\left\| \frac{d(a \circ \theta)}{dt}(t) \right\| < \frac{\delta}{t}.$$

Proof. Just note that the length of the path $a(t)$ is finite while the integral $\int_0^1 \delta t^{-1} dt$ is infinite for $\delta > 0$. \square

Let us start the proof of Proposition 2.

Proof of Proposition 2. Let $\mathcal{A} = \{A_i\}$ be an ε -flexible cocycle. First, we fix ε_1 sufficiently small so that $\varepsilon_1 + \text{dist}(\mathcal{A}_t) < \varepsilon$ holds. We also fix small $\eta > 0$. Precise choice of η is fixed at the end of this proof. By applying Lemma 3, we reparametrize the path $\mathcal{A}_t, t \in [-1, 1 - \eta]$ by a function $\theta: [0, +\infty[\rightarrow [-1, 1 - \eta]$ so that we have the following:

- For $t > 0$ we have the following inequality:

$$\left\| \frac{\partial A_{i, \theta(t)}}{\partial t} \right\| \leq \frac{\delta}{t},$$

- $\theta(t) = 0$ for $t \geq 1$,
- $\theta(t) = 1 - \eta$ near $t = 0$,
- There are $0 < t_3 < t_2 < t_1 < t_0 < 1$ such that:
 - We have $\theta(t) = -1$ for $t \in [t_3, t_0]$.
 - For $\{t_i\}$ we have the following inequalities:

$$t_2 < \lambda t_1, \quad t_3 < K^{-n} t_2 < t_1 K^n < t_0$$

where n is the period of cocycle, $K := \max\{\|A_{i,t}^{\pm 1}\|\} \geq 1$ and λ is the rate of the contraction of the homothety A_{-1} .

Then the announced cocycle $\mathcal{F} = \{f_i\}$ is defined by $f_i(x) := A_{i, \theta(\|x\|)}(x)$. Indeed, the Proposition 4, together with the first condition on θ implies the contraction property of \mathcal{F} . The last condition ensures that this cocycle (in the period) is an homothety of ratio λ on at least one fundamental domain, implying the retardable property. More precisely, by choosing $R_1 = t_3$, $R_2 = t_1$ and $R_3 = t_0$, we can check the retardable property. Note that by the choice of ε_1 , we can deduce that the diffeomorphism cocycle \mathcal{F} itself is an ε -perturbation of $\{A_i\}$. Note that Lemma 1 implies that its retarded cocycles are also ε -perturbation.

Furthermore, the cocycle \mathcal{F} is isotopic to \mathcal{A} through contracting cocycles which coincide with \mathcal{A} out of the unit ball, and whose periodic orbit has two distinct real positive eigenvalues: for that it is enough to change θ by $\theta_s(t) = s\theta(t)$ for $s \in [0, 1]$ (note that for every $s \in [0, 1]$ we can apply Proposition 4).

To finish the proof, it remains to perform an extra perturbation in a very small neighborhood of the periodic point for turning the weakest eigenvalue of $\mathcal{A}_{1-\eta}$ to be equal to 1 preserving the contracting property of the cocycle. We can see that if η is sufficiently small, then such a perturbation can be attained in the form of isotopy. More precisely, first we perform an perturbation so that the local dynamics along the periodic orbit exhibits an eigenvalue-one direction. Then we add another perturbation so that the central direction turns to be topologically attracting, keeping the eigenvalue.

Thus the proof is completed. \square

4.2 Realizing a path of linear cocycles as a diffeomorphisms cocycle: proof of Proposition 4

Let us start the proof of Proposition 4. We consider a relatively compact open subset $\mathcal{O} \subset \mathcal{O}_{n, \text{con}}$, (remember that $\mathcal{O}_{n, \text{con}}$ is the space of contracting linear cocycles of period n). Before starting the proof, let us have some auxiliary observations.

We put

$$K_{\mathcal{O}} := \max\{\|A_i^{\pm 1}\| \mid \mathcal{A} = \{A_i\}_{i \in \mathbb{Z}/n\mathbb{Z}} \in \mathcal{O}\} \geq 1,$$

that is, the bound of matrices and their inverse for the cocycles in \mathcal{O} .

The relative compactness of \mathcal{O} (compactness of the closure) implies that this bound is finite. The relative compactness of \mathcal{O} , together with the fact that each cocycle in \mathcal{O} is contracting, implies that their are uniformly contracting in the following sense:

Lemma 4. *Let \mathcal{O} be a relatively compact set of contracting linear cocycles. Then there is $k_{\mathcal{O}} > 0$ such that for every $\mathcal{A} = \{A_i\} \in \mathcal{O}$ and for every $i \in \mathbb{Z}/n\mathbb{Z}$ we have*

$$\|A_{i+k_{\mathcal{O}}-1} \circ \cdots \circ A_i\| < \frac{1}{2}.$$

Remark 6. *In fact, we will prove that the number δ announced by Proposition 4 only depends on ε , $K_{\mathcal{O}}$ and $k_{\mathcal{O}}$ and is independent of the period n and of the relatively compact set \mathcal{O} .*

Note that the relative compactness of \mathcal{O} also implies that $\overline{\mathcal{O}}$ does not contain any singular matrices. This fact, combined with compactness argument yields the following:

Lemma 5. *Given a relatively compact set $\mathcal{O} \subset \mathcal{C}_{n,\text{con}}$, there exists $\mu_{\mathcal{O}} > 0$ such that for every $\{A_i\} \in \mathcal{O}$, if $\{B_i\} \in M(2, \mathbb{R})^n$ (where $M(2, \mathbb{R})$ is a set of square matrices of size 2) satisfies $\|A_i - B_i\| < \mu_{\mathcal{O}}$ then we have $\{B_i\} \in \text{GL}(2, \mathbb{R})^n$.*

Remark 7. *Let $K > 1$. Then, the following set*

$$\mathcal{B}_K := \{\{A_i\} \in \text{GL}(2, \mathbb{R})^n \mid \max\{\|A_i^{\pm 1}\|\} \leq K\}$$

is a compact set. Thus, we can apply Lemma 5 to \mathcal{B}_K . We denote corresponding μ by μ_K .

Then, we prove the following.

Proposition 5. *Given $K \geq 1$, $\varepsilon_1 > 0$ and an integer $k > 0$ there exists $\delta > 0$ such that the following holds: Given any $n > 0$ and any path $\mathcal{A}_t = \{A_{i,t}\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, ($t \in [0, 1]$) satisfying:*

- $\|A_{i,t}^{\pm 1}\| < K$ for every i and t ,
- $\|A_{i+k-1,t} \circ \cdots \circ A_{i,t}\| < 1/2$ for every i and t ,
- $\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t}$.

Then the diffeomorphisms cocycle $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, where $f_i: \mathbb{R}^2 \times \{i\} \rightarrow \mathbb{R}^2 \times \{i+1\}$ are define as

$$f_i(x) := A_{i,\|x\|}(x)$$

satisfies:

- \mathcal{F} is a contracting diffeomorphisms contracting cocycle,
- For each point $(x, i) \in \mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$ and for every $0 \leq j \leq k-1$, we have

$$\|D\mathcal{F}^j(x, i) - A_{i+j-1,\|x\|} \circ \cdots \circ A_{i,\|x\|}\| < \varepsilon_1.$$

(Remember that \mathcal{F} is the diffeomorphism of total space $\mathbb{R}^2 \times \mathbb{Z}/n\mathbb{Z}$.)

Proof. First, note that the inequality in the second item implies the contracting property: the second item implies that $\|D\mathcal{F}^k\| < 1/2 + \varepsilon_1$, thus by exchanging ε_1 with a number smaller than $1/2$ if necessary, we obtain $\|D\mathcal{F}^k\| < 1$.

Let us start proving this inequality for $j = 1$. By direct calculation, for every i , we have

$$Df_i(x) = D(A_{i,\|x\|})(x, i) = A_{i,\|x\|} + \left(\frac{dA_{i,t}}{dt} \Big|_{t=\|x\|} \right) \otimes \left(\frac{\partial(\|x\|)}{\partial x} \right) (x),$$

Therefore, we have

$$\|Df_i(x) - A_{i,\|x\|}\| \leq C\|x\| \cdot \left\| \frac{dA_{i,\|x\|}}{dt} \right\|,$$

where C is a constant which does not depend on particular choice of other constants.

Then, we fix δ so that $\delta < \min\{\varepsilon_1, \mu_K\}/C$ holds (for the definition of μ_K see Lemma 5 and Remark 7). This guarantees that $\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t} < \min\{\varepsilon_1, \mu_K\}/(Ct)$. So we have:

$$\|Df_i(x) - A_{i,\|x\|}\| \leq \|x\| \cdot \frac{C\delta}{\|x\|} = C\delta < \min\{\varepsilon_1, \mu_K\},$$

A priori, each which implies the desired inequality for $j = 1$. Furthermore, this inequality, together with Lemma 5, shows that $Df_i : \mathbb{R}^2 \times \{i\} \rightarrow \mathbb{R}^2 \times \{i+1\}$ is a local diffeomorphism. This fact, combined with some topological observation, concludes that f_i is a diffeomorphism for every i .

Now we start the proof of inequality above for general $j < k$. The difficulty is the following: The differential $D\mathcal{F}^j(x, i)$ is the product

$$D\mathcal{F}^j(x, i) = D\mathcal{F}(\mathcal{F}^{j-1}(x, i)) \circ \cdots \circ D\mathcal{F}(x, i).$$

A priori, the distance between $\mathcal{F}^{j-1}(x, i)$ and $(x, i+j)$ can be very big. Thus the corresponding differentiations can be very different. Our strategy is to choose sufficiently small δ so that the matrix $A_{i+\ell, \|F^\ell(x, i)\|}$ remains almost equal to $A_{i+\ell, \|x\|}$, for $0 \leq \ell \leq k-1$.

We start from bounding $\|\mathcal{F}^\ell(x, i)\|$ for $0 \leq \ell < k-1$ as follows:

$$\frac{\|x\|}{K^k} \leq \|F^\ell(x, i)\| \leq K^k \|x\|,$$

where K is the uniform bound of the norms $\|A_{i,t}^{\pm 1}\|$. Then, a simple argument involving the mean value theorem implies:

Claim 2. *For any $\nu > 0$ there is $\delta > 0$ such that if $\left\| \frac{\partial A_{i,t}}{\partial t} \right\| \leq \frac{\delta}{t}$, for every $t > 0$, then for every $t > 0$, $i \in \mathbb{Z}/n\mathbb{Z}$, ℓ satisfying $0 \leq \ell < k$ and $s \in [K^{-k}t, K^kt]$ we have*

$$\|A_{i+\ell, t} - A_{i+\ell, s}\| < \nu.$$

Also a simple compactness argument together with the continuity of product of matrices shows the following.

Claim 3. *There is $\nu > 0$ so that for any matrices $\{B_{i,t}\}_{i \in \mathbb{Z}/n\mathbb{Z}, t \in [0,1]}$ with $\|B_{i,t} - A_{i,t}\| < 2\nu$, any $0 \leq j < k$ and for any $i, t, \varepsilon_1 > 0$ we have:*

$$\|B_{i+j-1, t} \circ \cdots \circ B_{i, t} - A_{i+j-1, t} \circ \cdots \circ A_{i, t}\| < \varepsilon_1.$$

Now we are ready for proving the inequality. We fix $\nu > 0$ given by Claim 3 and δ given by Claim 2. Then for every $0 \leq \ell < k$ we have

$$\begin{aligned} \|D\mathcal{F}(\mathcal{F}^\ell(x, i)) - A_{i+\ell, \|x\|}\| &\leq \|D\mathcal{F}(\mathcal{F}^\ell(x, i)) - A_{i+\ell, \|\mathcal{F}^\ell(x, i)\|}\| + \|A_{i+\ell, \|\mathcal{F}^\ell(x, i)\|} - A_{i+\ell, \|x\|}\| \\ &\leq \nu + \nu = 2\nu. \end{aligned}$$

The choice of ν implies the announced inequality

$$\|D\mathcal{F}^j(x, i) - A_{i+j-1, \|x\|} \circ \cdots \circ A_{i, \|x\|}\| < \varepsilon_1.$$

Thus the proof is completed. \square

5 Abundance of flexible periodic points

The aim of this section is the proof of Theorem 3 (and therefore of Theorem 2).

The proof contains two steps. The first step is showing that the hypotheses of Theorem 3 lead (up to arbitrarily small perturbations) to the coexistence of points with complex stable eigenvalues and points with a stable eigenvalue arbitrarily close to 1 in the same basic set. The second one is showing that such basic sets contain flexible points.

In this section, M denotes a smooth compact manifold endowed with a Riemannian metric.

5.1 Periodic points with arbitrarily weak stable eigenvalue

Let us start the proof of first step. We show the following.

Proposition 6. *Let $f \in \text{Diff}^1(M)$ with a hyperbolic periodic point p and an open neighborhood U of the orbit $\mathcal{O}(p)$ with the following property*

- *p has stable index equal to 2.*
- *there is a stable index 1 periodic point q with $\mathcal{O}(q) \subset U$.*
- *there are hyperbolic transitive basic sets $K \subset H(p, U)$ and $L \subset H(q, U)$ containing p and q , respectively, so that K, L form a C^1 -robust heterodimensional cycle in U (that is, the orbits of K and L are contained in U and there exists heteroclinic points between $W^u(K)$ and $W^s(L)$, $W^u(L)$ and $W^s(K)$ whose orbit is contained in U).*

then, for every $\nu > 0$ there are g arbitrarily C^1 -close to f having a periodic point p_2 with the following properties

- *p_2 has stable index 2 and is homoclinically related with p in U ;*
- *p_2 has a real stable eigenvalue $\lambda^{cs}(p_2)$ with $|\lambda^{cs}| \in [1 - \nu, 1)$;*
- *p_2 has the smallest Lyapunov exponent so that*

$$\chi^{ss}(p_2) \in [\inf\{\chi^{ss}(p), \chi^{ss}(q)\} - \nu, \sup\{\chi^{ss}(p), \chi^{ss}(q)\} + \nu],$$

- *the orbit of p_2 is ν -dense in the relative homoclinic class $H(p, U, f)$ for f .*

Sketch of proof of Proposition 6. The creation of periodic orbits with eigenvalues arbitrarily close to 1 inside a homoclinic class containing a robust heterodimensional cycle has been already done in [ABCDW]. The proof of Proposition 6 can be carried out in the similar fashion. So we only show the sketch of the proof of it.

We fix $\nu > 0$. First, note that K or L is non trivial since otherwise they cannot have robust heterodimensional cycle. Thus by performing perturbation by Hayashi's connecting lemma if necessary, we can assume that both of them are not trivial.

Recall that the homoclinic class of p is the closure of transverse homoclinic intersection, hence is the Hausdorff limit of an increasing sequence of hyperbolic basic sets. The corresponding fact is also true for relative homoclinic classes. Therefore one can choose a hyperbolic basic set $\tilde{K} \subset U$ whose Hausdorff distance with $H(p, U, f)$ is less than $\nu/10$. We can also choose an arbitrarily small perturbation f_0 of f and a periodic point $\tilde{p} \in \tilde{K}(f_0)$ where $\tilde{K}(f_0)$ is the hyperbolic continuation of \tilde{K} so that

- the Hausdorff distance between the orbit $\mathcal{O}(\tilde{p})$ and $H(p, U, f_0)$ is less than $\nu/5$,
- $|\chi^{ss}(\tilde{p}, f_0) - \chi^{ss}(p, f)| < \nu/10$.
- the Lyapunov exponent $\chi^{ss}(\tilde{p}, f_0)$ has multiplicity 1, that is, the restriction of the derivative to the stable plane has two distinct real eigenvalues (see section 2 of [ABCDW] or section 4 of [BCDG]).

If the perturbation f_0 is sufficiently close to f , then one still has a C^1 -robust heterodimensional cycle associated with $K(f_0)$ and $L(f_0)$ (and therefore $\tilde{K}(f_0)$ with $L(g_0)$). In other words, by replacing f with f_0 and ν with $\nu/2$, one may assume that

- the orbit of p is $\nu/2$ dense in $H(p, U, f)$
- p has two real distinct eigenvalues

In the same way, by changing q with another periodic point in L and performing an arbitrarily small perturbation of f , one may assume that q has a real weakest unstable eigenvalue.

Then, we perform second perturbation to construct a heterodimensional cycle between p to q in U as follows (see section 2 of [ABCDW]):

- as p and q belong to the same chain recurrence class, an arbitrarily small perturbation (using for instance the connecting lemma in [BC]) allows to create a transverse intersection between $W^u(q)$ and $W^s(p)$.
- as the C^1 -robust cycle persists under the first perturbation, p and q still belong to the same class so that an arbitrarily small perturbation, preserving the first intersection, allows to create a transverse intersection between $W^s(q)$ and $W^u(p)$.

Now [ABCDW] (see section 3 of [ABCDW], see also [BD2], [BDK]) tells us that by performing an arbitrarily small perturbation to the heterodimensional cycle as above, we can create a periodic point p_2 with the following property (we denote the perturbed diffeomorphism by g):

- p_2 has a stable index 2
- p_2 has a weakest stable eigenvalue λ^{cs} with absolute value $|\lambda^{cs}| = 1 - \nu < 1$
- the orbit of p_2 passes arbitrarily close to p_g (as a consequence $\mathcal{O}(p_2)$ will be $\nu/2$ dense in $H(p, U, f)$)
- $\chi^{ss}(p_2)$ is arbitrarily close to a convex sum of $\chi^{ss}(p, f)$ and $\chi^{ss}(q, f)$
- the unstable manifold of p_2 cuts transversely the stable manifold of p and the stable manifold of p_2 cuts transversely the unstable of q .

The last item implies that p_2 and p are robustly in the same chain recurrence class (since we can always find a pseudo-orbit from q to p following the robust heterodimensional cycle between K and L): a new arbitrarily small perturbation by connecting lemma in [BC] creates a transverse intersection between the stable manifold of p with the unstable of q , ending the proof. \square

5.2 Weak eigenvalues, complex eigenvalues, and flexible points

The aim of the remaining part of the section is the proof of the next proposition.

Proposition 7. *Given $C > 1$, $\chi < 0$ and $\varepsilon > 0$, there exists $\nu \in (0, 1)$ with the following property: Let $f \in \text{Diff}^1(M)$ be a diffeomorphism and Λ be an compact invariant hyperbolic basic set of f with stable index two such that $\|Df\|$ and $\|Df^{-1}\|$ are bounded by C over Λ .*

Suppose that Λ contains a periodic point q (of stable index two) with complex (non-real) stable eigenvalue and a point p having two distinct real stable eigenvalues so that

- *the smallest Lyapunov exponent of p is less than $\chi < 0$*
- *the stable eigenvalue with larger absolute value λ^{cs} satisfies $|\lambda^{cs}| \in (1 - \nu, 1)$.*

Then f admits arbitrarily small perturbations with ε -flexible points contained the continuation of Λ and the ε -neighborhood of $\mathcal{O}(x_L)$ contains $\mathcal{O}(p)$ in the Hausdorff topology.

This proposition, together with the previous Proposition 6 with standard genericity argument (involving the generic continuity of the homoclinic classes with respect to the Hausdorff distance) implies Theorem 3.

The main ingredient of the proof is the following fact: in a basic sets, given a set of finite number of periodic points, one may choose a periodic point which travels around these periodic points with the predetermined itinerary. By choosing a convenient itinerary, we can find a periodic point whose differential behaves in the way which is very close to what we want. Thus, by adding some perturbation, we can obtain the desired orbit. This technique has been formalized in [BDP] by the notion of *transition*.

5.3 Transitions on the periodic points

In this subsection we will extract some consequence from [BDP] which will be useful for us. For the proof of Lemma 6, see Lemma 1.9 of [BDP] (indeed, Lemma 6 is just the special case of Lemma 1.9 of [BDP]).

Lemma 6. *Suppose $f \in \text{Diff}^1(M)$ have hyperbolic basic set Λ with stable index 2. Let $T\Lambda = E^s \oplus E^u$ be the hyperbolic splitting (thus $\dim E^s = 2$). We fix coordinate on $\Lambda|_{E^s}$ so that and take the matrix representation of df . Let $x_1, x_2 \in \Lambda$ be two hyperbolic periodic saddle points of period π_i ($i = 1, 2$), respectively.*

Then, given $\varepsilon_2 > 0$, there exists two finite sequence of matrices in $\text{GL}(2, \mathbb{R})$ (T_i^j) ($j = 0, \dots, j_i - 1$) ($i = 1, 2$) (where j denotes the suffix, not the power of the matrix), with the following property:

For any $L = (l_1, l_2, l_3, l_4) \in \mathbb{N}^4$ satisfying $(l_1, l_2) \neq (l_3, l_4)$, there exists a periodic point $x_L \in \Lambda$ such that the following:

- *The period of x_L is $(l_1 + l_3)\pi_1 + (l_2 + l_4)\pi_2 + 2(j_1 + j_2)$.*
- *If $k = K$ with $0 \leq K < \pi_1 l_1$ or $k = l_1 \pi_1 + l_2 \pi_2 + j_1 + j_2 + K$ with $0 \leq K < \pi_1 l_3$, then $Df|_{E^s}(f^k(x_L))$ is ε_2 -close to $Df|_{E^s}(f^k(x_1))$.*
- *If $k = l_1 \pi_1 + j_1 + K$ with $0 \leq K < \pi_2 l_2$ or $k = (l_1 + l_3)\pi_1 + l_2 \pi_2 + 2j_1 + K$ with $0 \leq K < \pi_2 l_4$, then $Df|_{E^s}$ is ε_2 -close to $Df|_{E^s}(f^K(x_2))$.*
- *If $k = l_1 \pi_1 + K$ or $k = (l_1 + l_3)\pi_1 + l_2 \pi_2 + K$ with $0 \leq K < j_1$, then $Df|_{E^s}(f^k(x_L))$ is ε_2 -close to T_1^K .*
- *If $k = l_1 \pi_1 + l_2 \pi_2 + j_1 + K$ or $k = (l_1 + l_3)\pi_1 + (l_2 + l_4)\pi_2 + 2j_1 + j_2 + K$ with $0 \leq K < j_2$, then $Df|_{E^s}(f^k(x_L))$ is ε_2 -close to T_2^K .*

We put $T_i := \prod_{j=0}^{j_i-1} T_i^j$ and call them transition matrices.

Remark 8. *In the above Lemma, by adjusting L we can control the position of the orbit of x_L . More precisely, if we take l_1 or l_3 (resp. l_2 or l_4) very large, then x_L passes arbitrarily close to x_1 (resp. x_2). See [BDP] for detail.*

5.4 Rudimentary results from linear algebra

We collect two results from linear algebra, which will be used in the proof of Proposition 7.

First, we prove the following lemma.

Lemma 7. *Let $T \in \text{GL}_+(2, \mathbb{R})$ and Q be a contracting homothety (i.e., $Q = \lambda \text{Id}$ where λ satisfying $0 < \lambda < 1$). Then, given $\varepsilon > 0$ exists $h > 0$ such that the sequence of matrix (J_i) (resp. (L_i)) ($i = 0, \dots, h-1$) such that*

- *each J_i (resp. L_i) is ε -close to Q .*
- *The product $T(\prod_{i=0}^{h-1} J_i)$ (resp. $(\prod_{i=0}^{h-1} L_i)T$) is a contracting homothety.*

Proof. We only give the proof of the existence of (J_i) . The proof of (L_i) can be carried out similarly.

Let T , Q , and ε be given. First, given Q , we fix $\delta > 0$ such that the following holds: If $X \in \text{GL}_+(2, \mathbb{R})$ is δ -close to Id , then Q and XQ are ε -close. We can fix such δ because of the continuity of the multiplication.

For every $T \in \text{GL}_+(2, \mathbb{R})$, there is a continuous path $I(t)$ in $\text{GL}_+(2, \mathbb{R})$ such that $I(0) = T$ and $I(1) = \text{Id}$ (since $\text{GL}_+(2, \mathbb{R})$ is path-connected). Then, because of the compactness of the path, we can take a sufficiently large integer $m > 0$ such that the following holds: for every k ($0 \leq k < m-1$), $I((k+1)/m)(I(k/m))^{-1}$ is δ -close to the identity. We put $h = m$ and $J_k = I((k+1)/m) \cdot (I(k/m))^{-1} Q$. Since Q is a homothety, we have

$$\begin{aligned} T \prod_{k=0}^{h-1} (J_k) &= T \prod_{k=0}^{h-1} (I((k+1)/h) \cdot (I(k/h))^{-1} Q) \\ &= T Q^h \prod_{k=0}^{h-1} I((k+1)/h) \cdot (I(k/h))^{-1} \\ &= T Q^h \cdot I(1) \cdot (I(0))^{-1} = Q^h. \end{aligned}$$

This completes the proof. \square

Let us see the second Lemma. By $R(\theta)$ we denote the rotation matrix of angle θ , more precisely, we put

$$R(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

We need to check the following:

Lemma 8. *Let $1 > \lambda_1 > \lambda_2 > 0$. Then, for $0 < t < 1$, The matrix*

$$M(t) := R(-t) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R(t) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfies the following properties:

- For $0 \leq t < \pi/2$, $M(t)$ has two distinct positive contracting eigenvalues.
- For $t = \pi/2$, $M(t)$ is a homothetic contraction.

Proof. Both of items can be checked by calculating the characteristic polynomial of $M(t)$ directly. Indeed, it is given by $x^2 - \text{tr}(M(t))x + \det(M(t))$. Note that $\det(M(t))$ is equal to $(\lambda_1 \lambda_2)^2$ (independent of t). By a direct calculation, we can see that $\text{tr}(M(t))$ is equal to

$$(\lambda_1^2 + \lambda_2^2) \cos^2 t + 2\lambda_1 \lambda_2 \sin^2 t.$$

One can check that this value is monotone decreasing on $[0, \pi/2]$. Then, by a direct calculation we can check the desired properties of the path above. \square

5.5 Proof of Proposition 7

Let us start the proof of Proposition 7. For that we use Lemma 8, which explains the behavior of products of rotation matrices and diagonal matrices. To use it, we need to take convenient bases on tangent spaces of our basic set. First let us see how to fix such bases.

5.5.1 Diagonalizing coordinates

In the assumption of Proposition 7, the basic set Λ contains a periodic point p with distinct real eigenvalues: one eigenvalue is very close to ± 1 but the smallest Lyapunov exponent is bounded away from 0. In particular, the restriction of the differential $Df^\pi(p)$, where π is the period of p , to the stable plane is diagonalisable. We want to take the pair of eigenvectors as the basis of stable tangent directions along the orbit of p , but such a coordinate change may be big (with respect to the metric induced from the original Riemannian metric) and the seemingly small perturbations looked through the diagonalized coordinate can be very big. Thus, let us consider the size of such coordinate change.

To clarify our argument, we generalize the situation as follows. Let $A_i : V_i \rightarrow V_{i+1}$ ($i \in \mathbb{Z}/\pi\mathbb{Z}$) be a sequence of isomorphisms between the sequence of two dimensional Euclidean vector spaces V_i . Suppose the product $A_{\pi-1} \cdots A_0$ is diagonalizable. Thus, on each V_i , there are images of two eigenspaces U_i and W_i of dimension one. We assume that U_i is the strong contracting direction.

For each V_i , we fix an orthonormal basis $\langle e_{1,i}, e_{2,i} \rangle$ so that $U_i \wedge W_i$ and $(e_{1,i}) \wedge (e_{2,i})$ defines the same orientation. Then, let $B_i \in \text{SL}(2, \mathbb{R})$ be the matrix representing the change of the basis from the orthonormal one to the one having one unit vector on the most contracted eigendirection. More precisely, $B_i : V_i \rightarrow V_i$ is the (unique) matrix in $\text{SL}(2, \mathbb{R})$ satisfying $B_i(e_{1,i}) \in U_i$ and $\|B_i(e_{1,i})\| = 1$ (we take the matrix representation regarding $\langle e_{1,i}, e_{2,i} \rangle$ to be the standard basis). Up to multiplication by an orthogonal matrix (derives from the ambiguity of the choice of initial orthonormal basis), B_i is well defined. In particular, $\|B_i\|$ is well defined.

The value $\|B_i\|$ measures the angle between two eigendirections on V_i in the sense that it is a strictly decreasing function of the angle. The norm $\|B_i\|$ is equal to 1 when two eigendirections are orthogonal and diverges to $+\infty$ as the angle tends to 0 (this is easy to prove, so we omit it).

Now, we use the following Lemma from linear algebra:

Lemma 9. *For every $C_1 > 1$ and $\chi < 0$, there exists $\alpha > 0$ such that the following holds: Let $A_i : V_i \rightarrow V_{i+1}$ ($i \in \mathbb{Z}/\pi\mathbb{Z}$) be a sequence of isomorphisms between two dimensional Euclidean vector spaces with norms $\|A_i^\pm\| < C_1$. Suppose the product $A_{\pi-1} \cdots A_0$ is diagonalizable and the Lyapunov exponents of it satisfy the following:*

- *The smaller one is less than χ .*
- *The larger one is greater than $\chi/2$.*

then, there is $i \in \mathbb{Z}/\pi\mathbb{Z}$ on which the matrix of coordinate change B_i (defined as above) has a norm $\|B_i\|$ smaller than α .

We apply this Lemma 9 to our situation letting $C_1 = C$, χ given in the hypotheses of Proposition 7, $V_i = T_{f^i(p)}\Lambda|_{E^s}$ and $A_i = Df(f^i(x))|_{E^s}$. It implies that there is a constant α depending only on C_1, χ which there is at least a point $f^i(p)$ of the orbit of x where $\|B_i\| < \alpha$. Thus by replacing p with $f^i(p)$ we can assume $\|B_0\| < \alpha$. In other words, up to a conjugacy by matrices in $\text{SL}(2, \mathbb{R})$ bounded in norm by α , one may assume that $Df^\pi(x)$ is diagonal. This bounded change of coordinates induces a bounded change of the notion of perturbations (remember that, for matrices in $\text{SL}(2, \mathbb{R})$ the norm of the matrix and of its inverse are the same).

Thus, up to multiplication by some constant, we can assume that a δ -perturbation with respect to this coordinate is also a δ -perturbation to the orthonormal coordinate. So we fix some coordinate near $T_{f^i(p)}\Lambda|_{E^s}$ explained as above and continue the proof.

Let us prove the Lemma 9. First, note that a simple compactness argument shows the following.

Lemma 10. *Let $C_1 > 1$. Then, for every $\kappa > 1$ there exists $\tau > 0$ such that the following holds: For every $A \in \text{GL}(2, \mathbb{R})$ satisfying $\|A\|, \|A^{-1}\| < C_1$, if u, v are unit vector such that the angle between them is less than τ , then $1/\kappa < \|Au\|/\|Av\| < \kappa$.*

Then, let us prove Lemma 9.

Proof of Lemma 9. First, we fix $C_1 > 0$ and $\chi < 0$. Then, apply Lemma 10 for this C_1 letting $\kappa = \exp(-\chi/4)$ and fix τ . Let us see how we fix α . We fix α sufficiently large so that the following

holds: if $B \in \text{SL}(2, \mathbb{R})$ is the matrix representing the change of the basis with $\|P\| > \alpha$, then the corresponding angle is less than τ .

For such choice of α , we show the existence of good i where the corresponding matrix B_i has norm less than α . Suppose not, that is, every B_i has norm greater than α . This implies that on each $T_{f^i(x)}|_{E^s}$, the image of eigenvectors have angle less than τ . Then, Lemma 10 (using it inductively) implies that if u and v are eigenvectors in V_0 , then for each i we have $\exp((\chi/4)i) < \|(A_{i-1} \cdots A_0)(v)\|/\|(A_{i-1} \cdots A_0)(u)\| < \exp(-(\chi/4)i)$, but this contradicts to the hypotheses on the Lyapunov exponents of the first return map $A_{n-1} \cdots A_0$. \square

5.5.2 Creating homothety

Next lemma, inspired from [BDP] or [S] provides the announced homothety which is used to realize the division of perturbations:

Lemma 11. *Let $f \in \text{Diff}^1(M)$ and Λ be a non-trivial basic set of stable index 2. Suppose there exists a hyperbolic periodic point q which has a contracting complex eigenvalue. Then, there exists diffeomorphisms g , C^1 -arbitrarily close to f which has a periodic point r such that Whose differential to $dg^{\text{per}(r)}$ restricted to the stable direction is a contracting homothety. Furthermore, the support of the perturbation from f to g can be taken arbitrarily close to an orbit of a periodic point of f .*

The proof of Lemma 11 is essentially done in [BDP] (see Proposition 2.5 of [BDP]) or [S] (see Lemma 3.3 of [S]). Thus we only present the sketch of the proof.

Sketch of proof. The proof is a direct consequence of Lemma 7 and a variant of Lemma 6: The idea of Lemma 6 is that there are periodic orbits whose differential restricted to the stable direction is an arbitrarily small perturbation of a product of a fixed transition matrix T with an arbitrary power of the differential of the point q , which has a complex stable eigenvalue. Large powers of a matrix in $\text{GL}(2, \mathbb{R})$ with a complex eigenvalue admits perturbations so that the product is an homothety. Therefore, Λ contains periodic orbits whose stable derivative are arbitrarily small perturbation of the product of the transition matrix T by an arbitrary power of an homothety. Now Lemma 7 allows to perform a small perturbation along the orbit to cancel the fixed intermediate differentials. As a result, such periodic orbit has a homothety to the stable direction. \square

5.5.3 Creating flexible points: the proof of Proposition 7

Now we are ready for starting the proof of Proposition 7. Let us start.

Proof. Let $C > 1$ and $\chi < 0$ be given, and let K be the hyperbolic periodic points of stable index 2 containing periodic points p and q as is in the statement.

According to Lemma 9, we can fix coordinates on $T\mathcal{O}(p)|_{E^s}$ so that we may assume that:

- At the point p of period $\pi(p)$ the first return map has the form of diagonal matrix

$$Df^{\pi(p)}|_{E^s}(p) = P := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $|\lambda_1| \in [1 - \nu, 1)$ and the smaller Lyapunov exponent $(1/\pi(p)) \log |\lambda_2|$ is less than χ .

Furthermore, according to Lemma 11, by giving arbitrarily small perturbation whose support is away from some neighborhood of $\mathcal{O}(p)$ and changing r with q , we can assume that:

- At the point q , the first return map is a contracting homothety, that is, we have

$$Df^{\pi(q)}|_{E^s}(q) = Q := \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

where $\pi(q)$ denotes the period of q and $r \in (0, 1)$.

We choose ν so that multiplying Df by the homothety of ratio $(1 - \nu)^{-2}$ is a $\varepsilon/2$ perturbation. Note that the choice of ν can be determined only from the value of C and χ .

For proving Proposition 7, it remains to show that arbitrarily small perturbation of f may create ε -flexible point in Λ . We apply Lemma 6: there are transition matrices T_1, T_2 so that given any $L = (l_1, l_2, l_3, l_4)$ with $(l_1, l_2) \neq (l_3, l_4)$, we can find a periodic point x_L whose first return map of differential cocycle restricted to the stable direction admits a small perturbation so that the first return map is:

$$T_2 Q^{l_4} T_1 P^{l_3} T_2 Q^{l_2} T_1 P^{l_1}.$$

Indeed, by fixing ε_2 in the statement of Lemma 6 (which can be taken arbitrarily small), we can assume, by performing an ε_2 -perturbation, that the differential along x_L is indeed given by this matrix. Now, let us choose L so that the point x_L can be perturbed to an ε -flexible point.

5.5.4 First case: $T_i \in \text{GL}_+(2, \mathbb{R})$, and $\lambda_i > 0$

From here the proof varies depending on the signature of $\det(T_i)$ and λ_i . First, we consider the case $T_1, T_2 \in \text{GL}_+(2, \mathbb{R})$ and $\lambda_1, \lambda_2 > 0$. The case where some of them are in $\text{GL}_-(2, \mathbb{R})$ or $\lambda_i < 0$ can be treated very similarly. We will explain how one can adapt the proof in the other cases at the end of this proof.

Let us continue the proof. First, we apply Lemma 7 (letting $Q = Q$ and $T_1 = T$). Then we can find a sequence of matrices (L_i) ($i = 0, \dots, i_1 - 1$) such that each L_i is arbitrarily close to the homothety Q and $T_2(\prod_{i=0}^{i_1-1} L_i) = c_1 \text{Id}$. Similarly, we take (J_i) ($i = 0, \dots, i_2 - 1$) so that each J_i is arbitrarily close to Q and $(\prod_{i=0}^{i_2-1} J_i)T_1 = c_2 \text{Id}$ holds. We also fix an integer i_3 such that the multiplication of the rotation $R(\pi t/2i_3)$ to the matrix Q is an arbitrarily small perturbation of Q for every $t \in [-1, 1]$.

Then, let us fix $L = (l_1, l_2, l_3, l_4)$ as follows:

- $l_1 = l_3$ are sufficiently large integers so that $\mathcal{O}(x_L)$ contains $\mathcal{O}(\gamma_n)$ in its $\varepsilon/2$ -neighborhood with respect to the Hausdorff distance (see Remark 8).
- $l_2 \neq l_4$ and $l_2, l_4 > i_1 + i_2 + i_3$.
- The following inequality holds:

$$\mu_L := \exp \left(\frac{1}{\pi(x_L)} \cdot \log \left((c_1 c_2)^2 r^{l_2 + l_4 - 2(i_1 + i_2)} \lambda_1^{l_1 + l_3} \right) \right) > 1 - \nu.$$

We can take such L as follows: first take l_2, l_4 satisfying the second condition and later take l_1 and l_3 so large that the rest of the conditions are satisfied (remember that $\pi(x_L) = (l_1 + l_3)\pi(p) + (l_2 + l_4)\pi(q) + 2(j_1 + j_2)$, thus just by taking large l_1, l_3 we can obtain the inequality in the third condition).

First, we perform a preliminary perturbation along x_L so that the following holds (we denote the perturbed map also by f):

- For $k = K\pi(q) + l_1\pi(p) + j_1$ or $k = K\pi_2 + (l_1 + l_3)\pi(p) + l_2\pi(q) + 2j_1 + j_2$ ($K = 0, \dots, i_1 - 1$), we have

$$\prod_{i=0}^{\pi(q)-1} Df(f^{k+i}(x_L)) = L_k.$$

- For $k = K\pi(q) + l_1\pi(p) + j_1$ or $k = K\pi(q) + (l_1 + l_3)\pi(p) + l_2\pi(q) + 2j_1 + j_2$ ($K = i_0, \dots, i_1 + i_2 - 1$), we have

$$\prod_{i=0}^{\pi(q)-1} Df(f^i(x)) = J_k.$$

At this moment, by using the fact $(\prod J_i)T_1 = c_2 \text{Id}$, $T_2(\prod L_i) = c_1 \text{Id}$ and commutativity of homothetic transformation, we can check that the first return map of the derivative of f along x_L has the following form:

$$(c_1 c_2)^2 Q^{l_4 - i_1 - i_2} P^{l_3} Q^{l_2 - i_1 - i_2} P^{l_1} = (c_1 c_2)^2 Q^{l_2 + l_4 - 2(i_1 + i_2)} P^{l_1 + l_3}, \quad (\dagger)$$

in particular, x_L has two real contracting eigenvalues $(c_1 c_2)^2 r^{l_2 + l_4 - 2(i_1 + i_2)} \lambda_j^{l_1 + l_3}$, $j = 1, 2$.

We show that x_L is ε -flexible for this f by construction the path of linear cocycles for the flexibility. Recall that the path \mathcal{A}_t ($t \in [-1, 1]$) we need to construct is the one which joins the cocycle \mathcal{A}_0 induced by Df on the stable bundle to a cocycle \mathcal{A}_{-1} whose return map is a homothety, and to a cocycle \mathcal{A}_1 having an eigenvalue equal to 1.

First we build the path from \mathcal{A}_0 to \mathcal{A}_1 . For that we only need to multiply \mathcal{A}_0 along the orbit of x_L by a homothety of ratio $\mu_L^{-1} \cdot \exp(t)$. This gives a path of contracting cocycles between the original one and a homothetic one. Our second condition on the choice of L implies that the ratio of this homothety is always less than $(1 - \nu)^{-2}$ (and greater than one). Our choice of ν implies that the multiplication by such a homothety remains an ε -perturbation of \mathcal{A}_0 .

Now let us build the path \mathcal{A}_t for $t \in [-1, 0]$. For that purpose, we rewrite the product of the differential in the following way:

$$\begin{aligned} & T_2 \left(\prod_{k=0}^{i_1-1} L_k \right) Q^{l_4 - i_1 - i_2} \left(\prod_{k=0}^{i_2-1} J_k \right) T_1 P^{l_3} T_2 \left(\prod_{k=0}^{i_1-1} L_k \right) Q^{l_2 - i_1 - i_2} \left(\prod_{k=0}^{i_2-1} J_k \right) T_1 P^{l_1} \\ &= (c_1 \text{Id}) \underbrace{(Q^{i_3})}_{(**)} Q^{l_4 - i_1 - i_2 - i_3} (c_2 \text{Id}) P^{l_1} (c_1 \text{Id}) \underbrace{(Q^{i_3})}_{(*)} Q^{l_2 - i_1 - i_2 - i_3} (c_2 \text{Id}) P^{l_1}. \end{aligned}$$

In this product, we replace each of the homothetic matrix Q in the first Q^{i_3} (indicated by $(*)$) by $R(\frac{\pi t}{2i_3}) \circ Q$ and the ones in the second one (indicated by $(**)$) by $R(\frac{-\pi t}{2i_3}) \circ Q$. By the choice of i_3 , this perturbation can be done very small for every $t \in [-1, 0]$, especially, it can be done with size less than ε . The effect of this perturbation on the product matrix is to replace the whole product in (\dagger) to

$$(c_1 c_2)^2 Q^{l_2 + l_4 - 2(i_1 + i_2)} R\left(-\frac{\pi}{2}t\right) P^{l_1} R\left(\frac{\pi}{2}t\right) P^{l_3}.$$

Now Lemma 8 ensures (remember $l_1 = l_3$) that the product matrix of \mathcal{A}_t in the period has two different real contracting positive eigenvalues for $t \neq -1$ and is a homothety for $t = -1$.

Thus combining these two paths, we have completed the proof. \square

5.5.5 Other cases: matrices in $\text{GL}_-(2, \mathbb{R})$ or negative eigenvalues

Finally, let us consider the case where some of the signs are negative.

First, in the case where λ_1 or λ_2 are negative, we just need to take l_1, l_3 to be even numbers: this replaces the matrix P by P^2 everywhere in the proof, and the proof works identically.

In the case where one but only one of the transition T_i , say T_1 for instance, reverses the orientation. Then [BDP] allows us to consider the matrix $\tilde{T}_1 = T_1 P^i T_2 Q^j T_1$ as a new transition substituting T_1 (keeping T_2 unchanged as the other transition matrix), and now \tilde{T}_1, T_2 both belongs to $\text{GL}_+(2, \mathbb{R})$.

It remains the case where T_1 and T_2 are both orientation reversing. In this case, we apply Lemma 7 to the matrix

$$T_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which provides a sequence of matrices (L_i) ($i = 0, \dots, i_1 - 1$) arbitrarily close to the homothety Q so that

$$T_2 \left(\prod_{i=0}^{i_1-1} L_i \right) = (c_1 \text{Id}) \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly, we take (J_i) ($i = 0, \dots, i_2 - 1$) arbitrarily close to Q so that

$$\left(\prod_{i=0}^{i_2-1} J_i\right) T_1 = (c_2 \text{Id}) \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the proof works identically, just noticing that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an involution which commutes with P and Q .

Remark 9. *In the proof, we can assume that the first return map of differential to the unstable direction is also orientation preserving by adjusting the number of l_i if necessary.*

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